BAL BHARATI PUBLIC SCHOOL,NAVI MUMBAI
SESSION 2020-2021

MATHEMATICS

CLASS : XII
PART: I

CHAPTER:2

INVERSE TRIGONOMETRIC FUNCTIONS

- The **domain** of a function is the list of all possible inputs (*x*-values) to the function.

- The **range** of a function is the list of all possible outputs (*y*-values) of the function.

- Graphically speaking, the domain is the portion of the *x*-axis on which the graph casts a shadow.

- Graphically speaking, the range is the portion of the *y*-axis on which the graph casts a shadow.

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \to [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \to [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{ x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{ x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{ x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

Inverse Function

If y = f(x) and x = g(y) are two functions such that f (g(y)) = y and g (f(y)) = x, then f and y are said to be inverse of each other

i.e., $g = f^{-1}$

IF y = f(x), then $x = f^{-1}(y)$

Inverse Trigonometric Functions

If $y = \sin X^{-1}$, then $x = \sin^{-1} y$, similarly for other trigonometric functions.

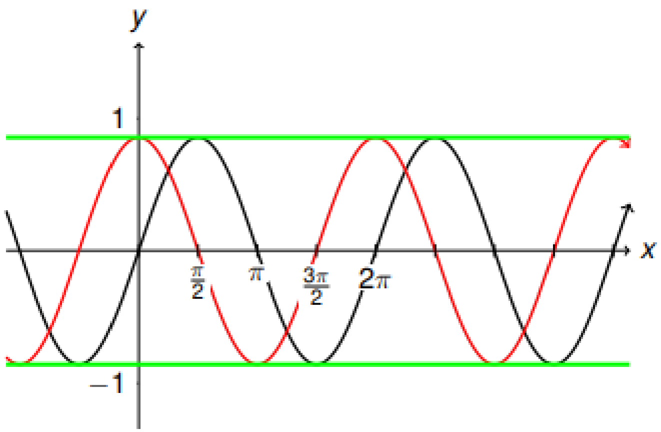
This is called inverse trigonometric function .

Now, $y = \sin^{-1}(x)$, $y \in [\pi / 2 , \pi / 2]$ and $x \in [-1,1]$.

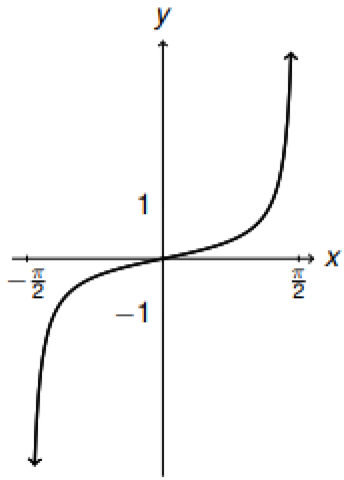
(i) Thus, $\sin^{-1}x$ has infinitely many values for given $x \in [-1, 1]$.

(ii) There is only one value among these values which lies in the interval $[\pi / 2 , \pi / 2]$. This value is called the principal value.

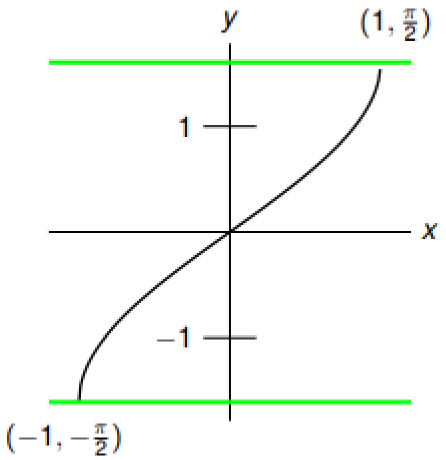
$y = \sin x \qquad y = \cos x$



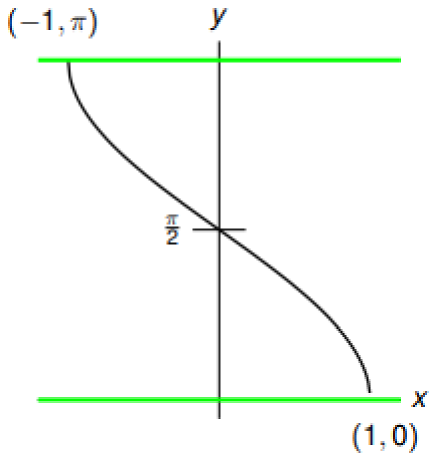
$y = \tan x$



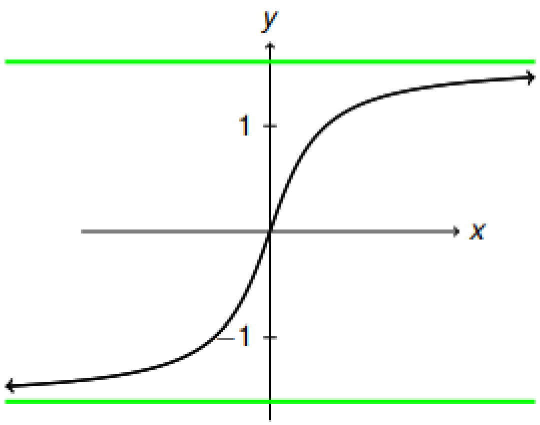
$y = \sin^{-1} x$



$y = \cos^{-1} x$



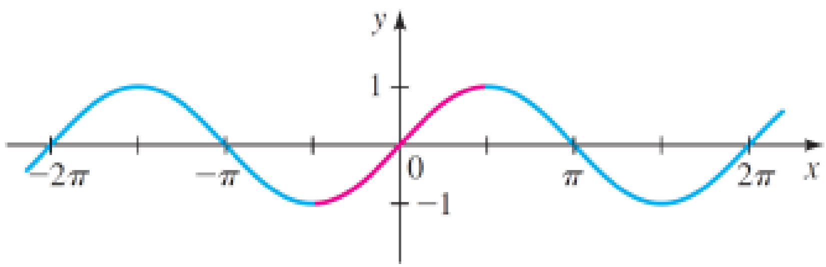
$y = \tan^{-1} x$



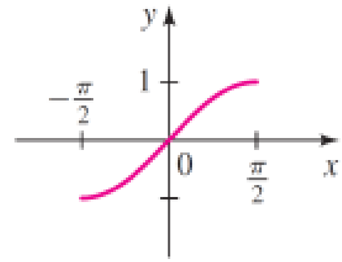
Inverse Trigonometric Functions and Their Graphs

DEFINI se sine function, denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

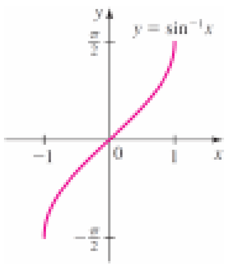
$$\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



$y = \sin x$

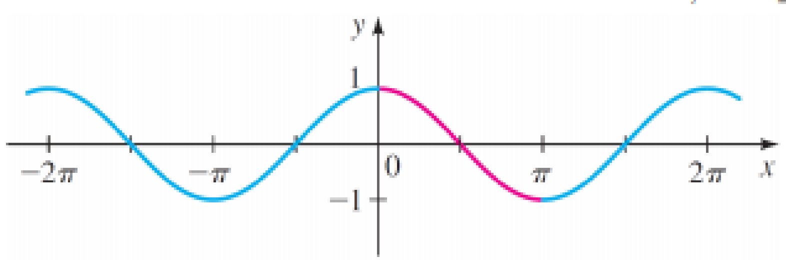


$y = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

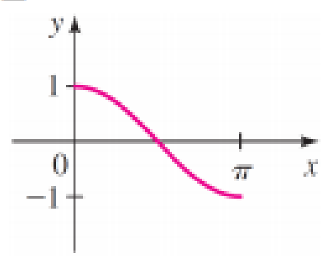


DEFINITION: The **inverse cosine function,** denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

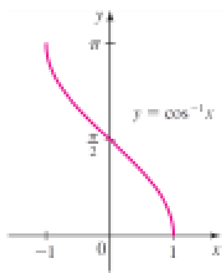
$$\cos x, \quad 0 \leq x \leq \pi$$



$y = \cos x$

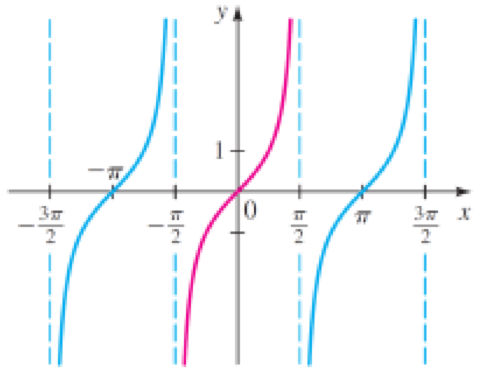


$y = \cos x,\ 0 \leq x \leq \pi$

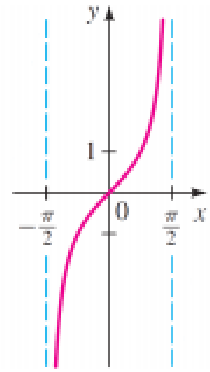


DEFINITION: The **inverse tangent function,** denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

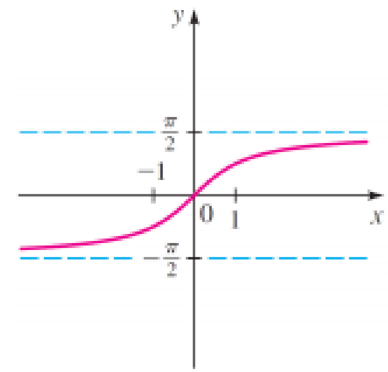
$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$y = \tan x$



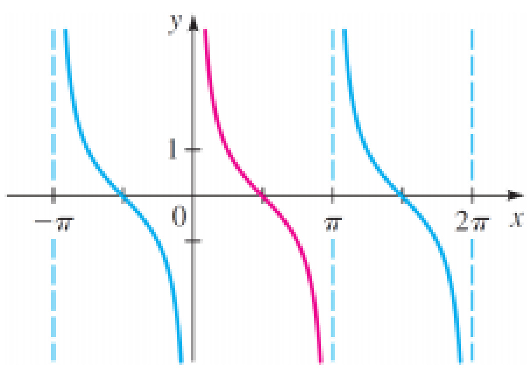
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



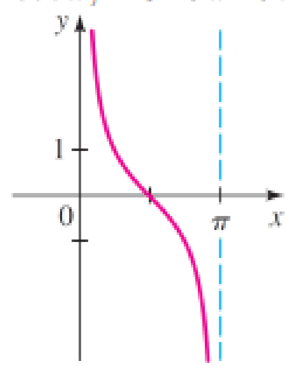
$y = \tan^{-1} x$

DEFINITION: The **inverse cotangent function,** denoted by $\cot^{-1} x$ (or $\operatorname{arccot} x$), is defined to be the inverse of the restricted cotangent function

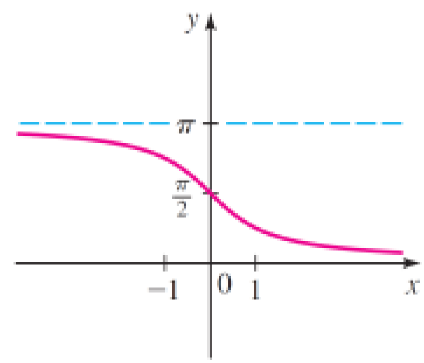
$$\cot x, \quad 0 < x < \pi$$



$y = \cot x$



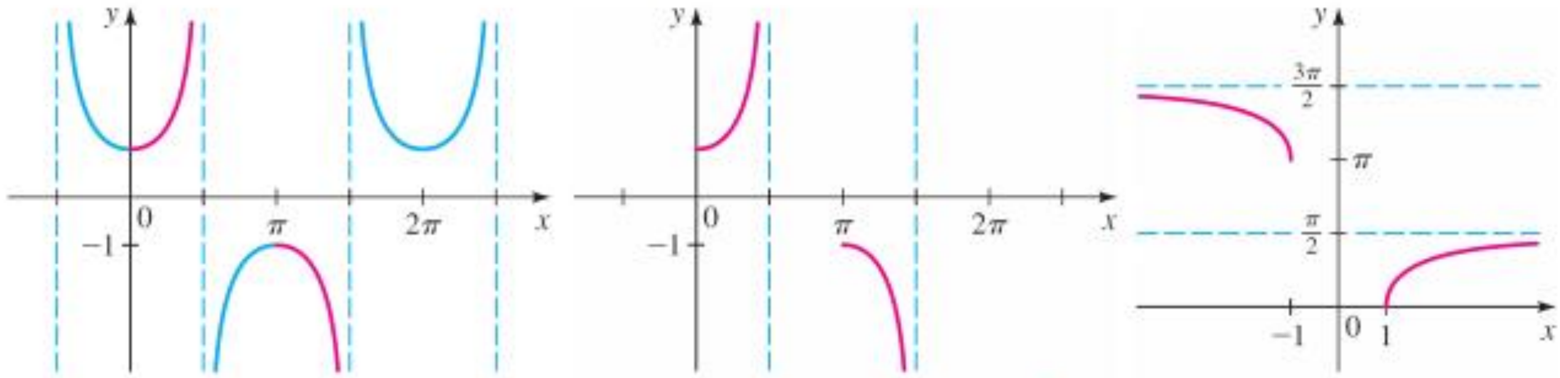
$y = \cot x,\ 0 < x < \pi$



$y = \cot^{-1} x$

DEFINITION: The **inverse secant function,** denoted by $\sec^{-1} x$ (or $\operatorname{arcsec} x$), is defined to be the inverse of the restricted secant function

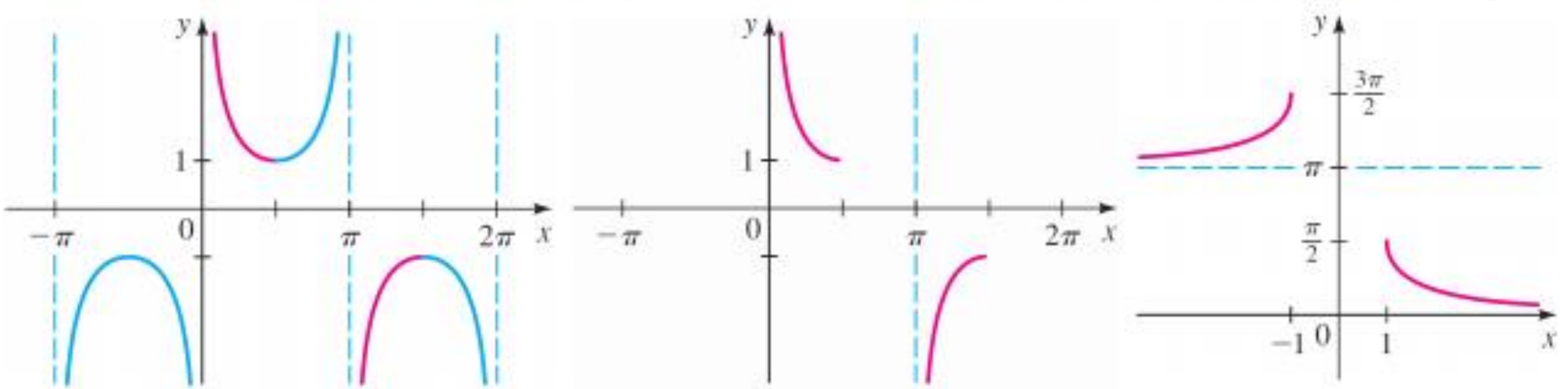
$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad [\text{or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks}]$$



$y = \sec x$ $\qquad$ $y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$ $\qquad$ $y = \sec^{-1} x$

DEFINITION: The **inverse cosecant function,** denoted by $\csc^{-1} x$ (or $\operatorname{arccsc} x$), is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad [\text{or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks}]$$



$y = \csc x$ $\qquad$ $y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$ $\qquad$ $y = \csc^{-1} x$

Function	Domain	Range
$y = sin(x)$	$-\infty < x < \infty$	$-1 \leq y \leq 1$
$y = cos(x)$	$-\infty < x < \infty$	$-1 \leq y \leq 1$
$y = tan(x)$	$x \neq \ldots -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$	$-\infty < y < \infty$
$y = sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

	Function	Domain	Range (Principal value)
1.	$y = \sin^{-1} x$	[−1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	[−1, 1]	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \operatorname{cosec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

EXAMPLE:1

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\sin^{-1}\left(-\frac{1}{2}\right)=y.$ Then $\sin y$ = $-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is

$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}.$

EXAMPLE:2

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\tan^{-1}(1)=x$. Then, $\tan x=1=\tan\frac{\pi}{4}$.

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y$. Then, $\cos y=-\frac{1}{2}=-\cos\left(\frac{\pi}{3}\right)=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z$. Then, $\sin z=-\frac{1}{2}=-\sin\left(\frac{\pi}{6}\right)=\sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

Property 1

i. $\sin^{-1}(1/x) = \text{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$

ii. $\cos^{-1}(1/x) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$

iii. $\tan^{-1}(1/x) = \cot^{-1}x$, $x > 0$

Proof : $\sin^{-1}(1/x) = \text{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$,

Let $\sin^{-1} x = y$

i.e. $x = \text{cosec } y$

$\frac{1}{x} = \sin y$

$\sin^{-1} \frac{1}{x}) = y$

$\sin^{-1} \frac{1}{x}) = cosec^{-1}x$

$\sin^{-1}(\frac{1}{x}) = cosec^{-1}x$

Hence, $\sin^{-1} \frac{1}{x} = cosec^{-1}x$ where, $x \geq 1$ or $x \leq -1$.

Property 2

i. $\sin^{-1}(-x) = -\sin^{-1}(x), \quad x \in [-1,1]$

ii. $\tan^{-1}(-x) = -\tan^{-1}(x), \quad x \in R$

iii. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x), |x| \geq 1$

Proof: $\sin^{-1}(-x) = -\sin^{-1}(x), \quad x \in [-1,1]$

Let, $\sin^{-1}(-x) = y$

Then $-x = \sin y$

$x = -\sin y$

$x = \sin(-y)$

$\sin^{-1} = \sin^{-1}(\sin(-y))$

$\sin^{-1} x = y$

$\sin^{-1} x = -\sin^{-1}(-x)$

Hence, $\sin^{-1}(-x) = -\sin^{-1}$ $x \in [-1,1]$

Property 3

i. $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

ii. $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \geq 1$

iii. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

Proof : $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

Let $\cos^{-1}(-x) = y$

$\cos y = -x \quad x = -\cos y$

$x = \cos(\pi - y)$

Since, $cos\pi - q = -cosq$

$\cos^{-1} x = \pi - y$

$\cos^{-1} x = \pi - \cos^{-1} - x$

Hence, $\cos^{-1} -x = \pi - \cos^{-1} x$

Property 4

i. $\sin^{-1}x + \cos^{-1}x = \pi/2, x \in [-1,1]$

ii. $\tan^{-1}x + \cot^{-1}x = \pi/2, x \in R$

iii. $\text{cosec}^{-1}x + \sec^{-1}x = \pi/2, |x| \geq 1$

Proof : $\sin^{-1}x + \cos^{-1}x = \pi/2, x \in [-1,1]$

Let $\sin^{-1} x = y$ or $x = \sin y = \cos(\frac{\pi}{2} - y)$

$\cos^{-1} x = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - y\right)\right)$

$\cos^{-1} x = \frac{\pi}{2} - y$

$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

$\sin^{-1} + \cos^{-1} x = \frac{\pi}{2}$

Hence, $\sin^{-1}x + \cos^{-1}x = \pi/2, x \in [-1,1]$

Property 5

i. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}((x+y)/(1-xy))$, $xy < 1$.

ii. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}((x-y)/(1+xy))$, $xy > -1$.

Proof : $\tan^{-1}x + \tan^{-1}y = \tan^{-1}((x+y)/(1-xy))$, $xy < 1$.

Let $\tan^{-1} x = A$

And $\tan^{-1} y = B$

Then, $\tan A = x$

$\tan B = y$

Now, $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$

$\tan(A + B) = \frac{x+y}{1-xy}$

$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = A + B$

Hence, $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} x + \tan^{-1} y$

Property 6

i. $2\tan^{-1}x = \sin^{-1}(2x/(1+x^2)), |x| \le 1$

ii. $2\tan^{-1}x = \cos^{-1}((1-x^2)/(1+x^2)), x \ge 0$

iii. $2\tan^{-1}x = \tan^{-1}(2x/(1-x^2)), -1 < x < 1$

Proof : $2\tan^{-1}x = \sin^{-1}(2x/(1+x^2)), |x| \le 1$

Let $\tan^{-1}x = y$ and $x = \tan y$

Consider RHS. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= \sin^{-1}\left(\frac{2\tan y}{1+\tan^2 y}\right)$

$= \sin^{-1}(\sin 2y)$

Since, $\sin 2\theta = 2\tan\theta/(1+\tan^2\theta)$,

$= 2y$

$= 2\tan^{-1}x$ which is our LHS

Hence $2\tan^{-1}x = \sin^{-1}(2x/(1+x^2)), |x| \le 1$

EX:1 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x, \; -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \; \frac{1}{\sqrt{2}} \le x \le 1$

SOL:

(i) Let $x = \sin\theta$. Then $\sin^{-1}x = \theta$. We have

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= 2\sin^{-1}x$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$

EX:2 Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$

SOL:

By property 5 (i), we have

$$\text{L.H.S.} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$$

EX:3

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right),\ 0 < x < \pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \tan^{-1}\left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy} = \tan^{-1}x - \tan^{-1}y\right]$$

$$= \frac{\pi}{4} - x$$

THANK YOU